Grading guide, Pricing Financial Assets, August 2019

1. Let the price of a traded financial instrument, S, be modelled (under the probability measure $\mathbb{P}$) by the geometric Brownian motion

$$dS = \mu S dt + \sigma S dz$$

where μ and $\sigma > 0$ are constants, and z is a Brownian motion.

(a) Describe the qualitative characteristics of this model, and discuss its possible shortcomings as a model of a stock price.

(b) Assume a constant risk free rate of r, and that the instrument pays a continuous dividend stream of δ proportional to the price S. What will the drift rate of the price be under the standard risk neutral probability measure ($\mathbb{Q}$) and a no-arbitrage assumption?

(c) Consider a derivative on S with value V equal to S^n. Use Ito's lemma to find the process followed by V. Is the volatility of V higher, if n is higher?

Solution:

(a) The issue that the model assumes no dividends aside (it is dealt with below), the discussion should cover

- The model implies continuous sample paths whereas stock prices can make large instantaneous changes (and in any case prices are only quoted in certain tick sizes).
- The model is Markov (which can be seen as a form of weak informational efficiency, cf. Hull (8ed, section 13.1-2))
- The model implies constant volatility, where we in practise observe marked (and sticky) changes in volatility
- The model implies normally distributed returns (whereas empirical stylized facts point to negative skewness and some excess kurtosis - not really discussed in Hull).

(b) The drift rate under $\mathbb{Q}$ should be $r - \delta$ so as to make the total return of holding the stock equal to the risk free interest rate (cf. Hull (8ed, section 16.3)).

(c) An application of Ito's lemma gives

$$\begin{aligned} dV &= (\tfrac{1}{2}\sigma^2 S^2 n(n-1)S^{n-2} + \mu S n S^{n-1})dt + nS^{n-1}\sigma S dz \\ &= (n\mu + \tfrac{1}{2}n(n-1)\sigma^2)V dt + n\sigma V dz \end{aligned}$$

So this is also of a geometric Brownian form. And it can be seen that volatility is increasing in n. This is Practice Question 13.10 in Hull (8ed); note that it is not the same as the value of a derivative that pays of S_T^n at time T, cf. Practice Question 14.12 in Hull (8ed).

2. The HJM-model describes the simultaneous evolution of the term structure of interest rates. Let the evolution of instantaneous forward rates contracted at t for time T be described by the Ito-process

$$df(t,T) = m(t,T,\Omega)dt + s(t,T,\Omega)dz$$

where Ω is a set of state variables.

(a) Under certain conditions we have the following no-arbitrage condition for the drift term:

$$m(t,T,\Omega) = s(t,T,\Omega)\int_t^T s(t,\tau,\Omega)d\tau$$

Comment on this result, and in particular explain under which probability measure it is derived.

(b) The short rate in this model is in general non-Markov. Explain what this means, and why it is a complication for implementation.

(c) As a special case let $s(t, T, \Omega)$ be a constant s. Derive the process followed by forward rates. Comment on the distribution of the forward rates.

Solution:

(a) What should be commented is that the volatility barring arbitrage determines the drift rate, and that the result presented is derived under the traditional risk neutral probability measure $\mathbb{Q}$, cf. Hull (8ed, pp. 716-7).

(b) As given in the lecture notes a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a process $X_t : \Omega \mapsto \mathbb{R}^n$ is Markov if

$$\mathbb{P}(X_{t_n} \leq x_n | X_{t_{n-1}} = x_{n-1}, \ldots, X_{t_1} = x_1) = \tag{1}$$

$$\mathbb{P}(X_{t_n} \leq x_n | X_{t_{n-1}} = x_{n-1}), \forall n, \forall t_1 < \ldots < t_n. \tag{2}$$

Hull has a less formal definition in section 13.1 (of the 8th edition).

I.e. if the short rate is *not* Markov we can not in general model the evolution of rates without including the full history of rates. In practise this means that we cannot e.g. use a recombining interest rate tree, which adds a substantial computational burden if more than a few time steps are to be simulated, cf. Hull (8ed, section 31.1).

(c) By integration we find for this simple model that

$$df(t, T) = s^2(T - t)dt + sdz$$

(this is not derived in the syllabus). This means that the changes to the forward rates, and thus the forward rates, will be normally distributed, which in particular means a positive probability for negative interest rates. It can be noted - but this is not directly in the syllabus (it is in Hull (8ed, Practice Question 31.3)) - that this special case is equivalent to the Ho-Lee-model.

3. (a) Consider a derivative with value $V(S, t)$ as some function of the current stock price S and time t (and further implicit parameters). Define the Delta, Gamma and Theta of the derivative. What can they be used for?

(b) Assume that the stock pays no dividends before time T, and that there is a constant risk free interest rate of r. Let $c(S, K, T, r)$ and $p(S, K, T, r)$ be the price at time $t = 0$ of a European call and a European put, respectively, on the stock with the same strike K and expiry T. Derive the call-put-parity.

(c) Use the call-put-parity to find a relationship between the Deltas of the call and put. Repeat this for Gamma and Theta, respectively.

(d) Suppose a portfolio of the stock and/or derivatives of that stock is Delta-neutral, and that there are no arbitrage possibilities. Let the value of the portfolio be $\Pi(S, t)$. Use the Black-Scholes-Merton PDE to characterise the relation between the Theta and Gamma of the portfolio.

Solution:

(a) The Delta and Theta of the derivative is the partial derivative of the value V with respect to the underlying stock price and the calendar time parameter t , respectively. Gamma is the second order derivative of the value with respect to the stock price (Cf. Hull (8ed) p.380ff). The Greeks above can be used to measure economic risks of a portfolio including options, and to manage and possibly hedge part of the risks.

(b) By considering the payoff at maturity we see that a portfolio with long European call and a short European put with strikes K and expiries T has the same payoff as a forward on the stock with forward price K and maturity T, thus also (under a no-arbitrage assumption)

$$c(S_0, K, T, r) - p(S_0, K, T, r) = S_0 - P\left(0, T\right) K$$

where here $P\left(0, T\right) = \exp(-rT)$.

(c) By considering the call-put-parity you immediately get for the Deltas

$$\Delta_c = 1 + \Delta_p$$

and using this for the gammas

$$\Gamma_c = \Gamma_p$$

For Theta we get

$$\Theta_c + rK\mathrm{e}^{-rT} - \Theta_p = 0$$

(using that the derivative with respect to calender time is minus the derivative with respect to the expiry date).

(d) For a portfolio with value $\Pi(S, t)$ dependent on a non-dividend-paying stock we have by no arbitrage the Black-Scholes-Merton PDE:

$$\frac{\partial \Pi}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \tfrac{\partial^2 \Pi}{\partial S^2} + rS\frac{\partial \Pi}{\partial S} - r\Pi = 0$$

or

$$\Theta + \tfrac{1}{2}\sigma^2 S^2 \Gamma + rS\Delta = r\Pi$$

When delta is zero you have

$$\Theta + \tfrac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

One interpretation from this is that a delta-neutral, positive-gamma portfolio barring arbitrage will have (for small r or small Π) a negative theta (i.e. time-decay), cf. Hull (8ed, section 18.7).